A particle method for solving Fredholm equation of the first kind

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Janvier 2026

General presentation

- inverse problems (Fredholm equation) : $h(y) = \int_{\mathbb{X}} f(x) g(y \mid x) \mathrm{d}x \qquad \forall y \in \mathbb{Y}$
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- EM(S) algorithm $f_{n+1}(x) = \int_{\mathbb{X}} K(x', x) f_n(x') \int_{\mathbb{Y}} \frac{g(y \mid x') h(y)}{\int_{\mathbb{X}} f_n(z) g(y \mid z) \mathrm{d}z} \, \mathrm{d}y \, \mathrm{d}x'$
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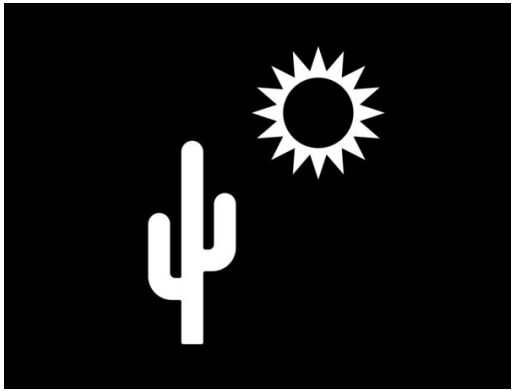
- Feynman-Kac formalism

$f(x)$  $g(y|x)$  $h(y)$

Feynman-Kac formalism of EMS

$$M_{n+1}\left((x_{n+1}, y_{n+1}) \mid (x_n, y_n)\right) = K(x_n, x_{n+1})h(y_{n+1})$$

$$G_n(x_n, y_n) = \frac{g(y_n \mid x_n)}{\int_{\mathbb{X}} \eta_n|_{\mathbb{X}}(z) g(y_n \mid z)\, \mathrm{d}z}$$

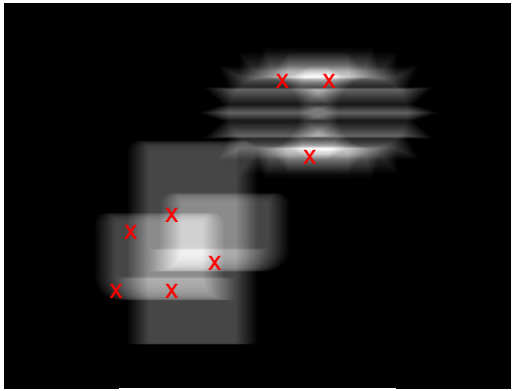
$$\eta_{n+1}(x, y) = \int_{\mathbb{X}\times\mathbb{Y}} \eta_n(x, y) M_{n+1}((x, y)|(x', y'))G(x', y')dx'dy'$$

By augmenting the state space to “sharp” and “blurred” image, we obtain a Feynman-Kac recursion

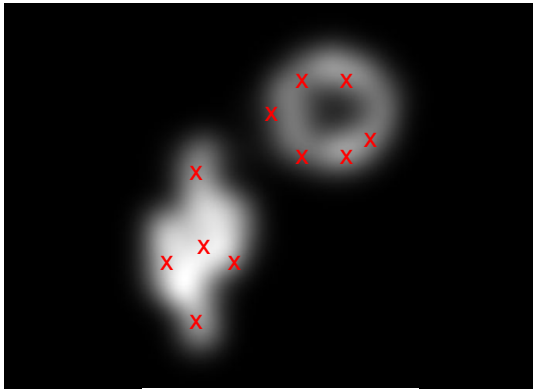
The particle approximation

$$(X_n^i, Y_n^i)_{1 \leq i \leq N}$$

$$\eta_n \to \eta_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_n^i, Y_n^i)}$$



$(Y_n^i)_{1 \leq i \leq N} \sim h$



$(X_n^i)_{1 \leq i \leq N} \sim f_n$

Approximation for potentials

- Particles $(X_n^i, Y_n^i)_{1 \leq i \leq N}$ $\eta_n \to \eta_n^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_n^i, Y_n^i)}$ $(f_n \to f_n^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_n^i}, h \to h^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_n^i})$
- Approximate potentials

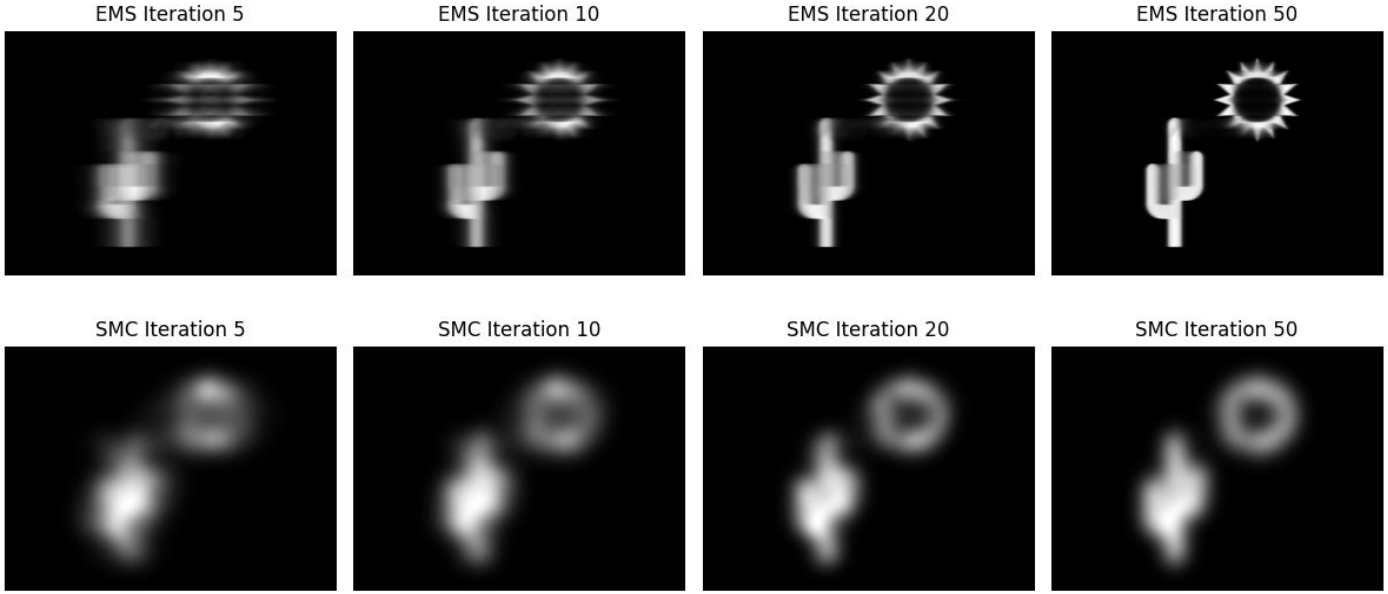
$$G_n(x_n, y_n) = \frac{g(y_n|x_n)}{\int_{\mathbb{X}} f_n(z) g(y_n|z) dz} \to G_n^N(x_n, y_n) = \frac{g(y_n|x_n)}{\frac{1}{N} \sum_{i=1}^{N} g(y_n|X_n^i)}$$

Second integral approximation

The quantity $$\int_{\mathbb{Y}} \frac{g(y|X_n^i)}{\frac{1}{N}\sum_{k=1}^{N} g(y|X_n^k)} h(y)dy$$

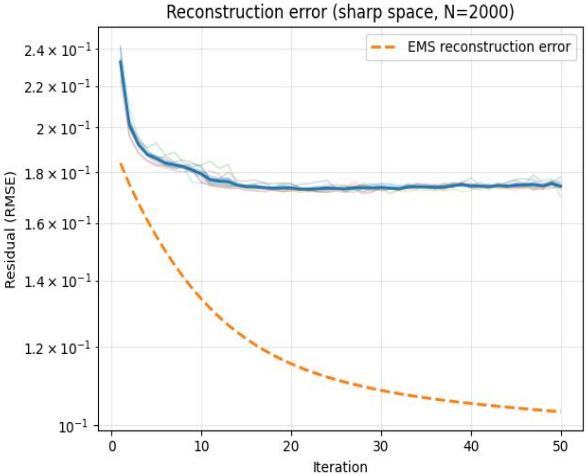
is approximated with Monte-Carlo by sampling a number M of elements Y with density h $(Y_n^i)_{1\leq i\leq N} \sim h$

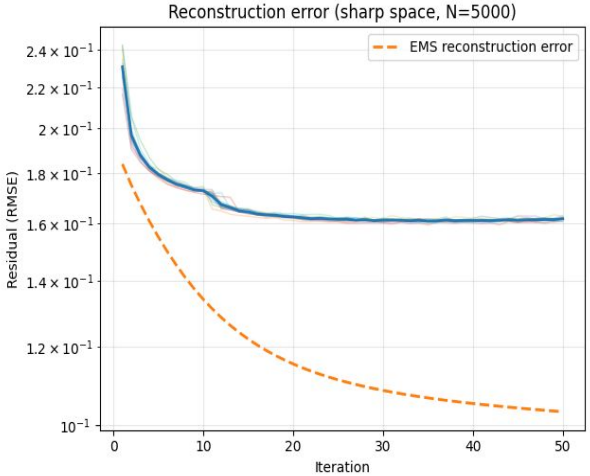
First results

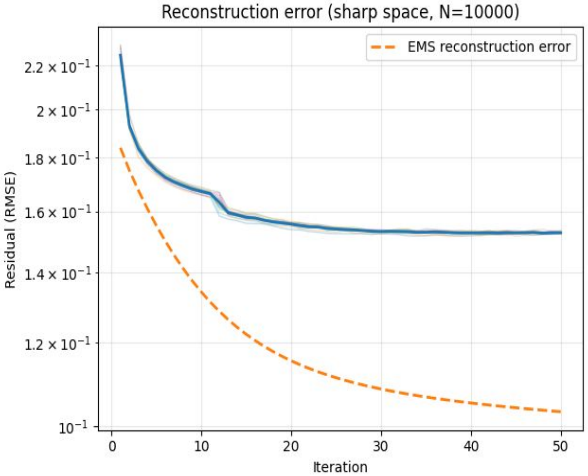


quick fix-point convergence

First results

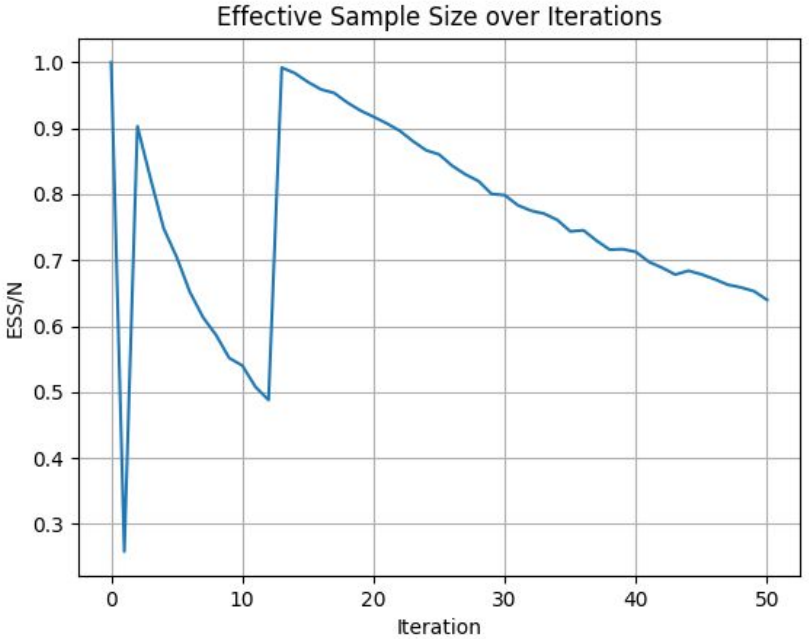






ESS

multinomial resampling if ESS < N/2

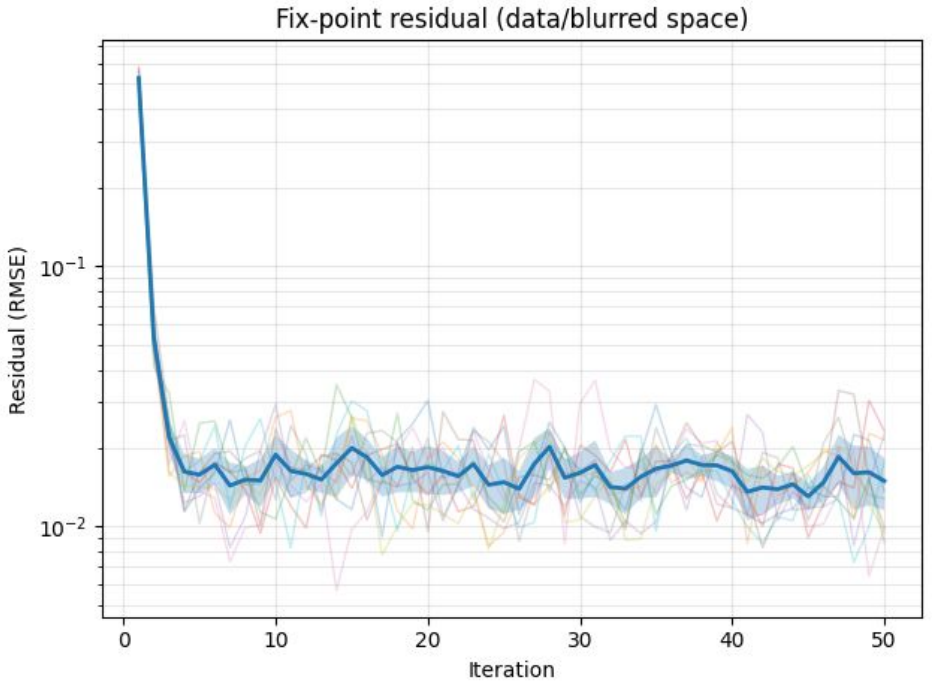


Fix-point convergence

Proposition 1: EMS has a fix-point f^*

We plot $\mathrm{RMSE}(\hat{h}^N_{n+1} - \hat{h}^N_n)$

Plateaus quickly (not to 0 because of Monte-Carlo approximation)



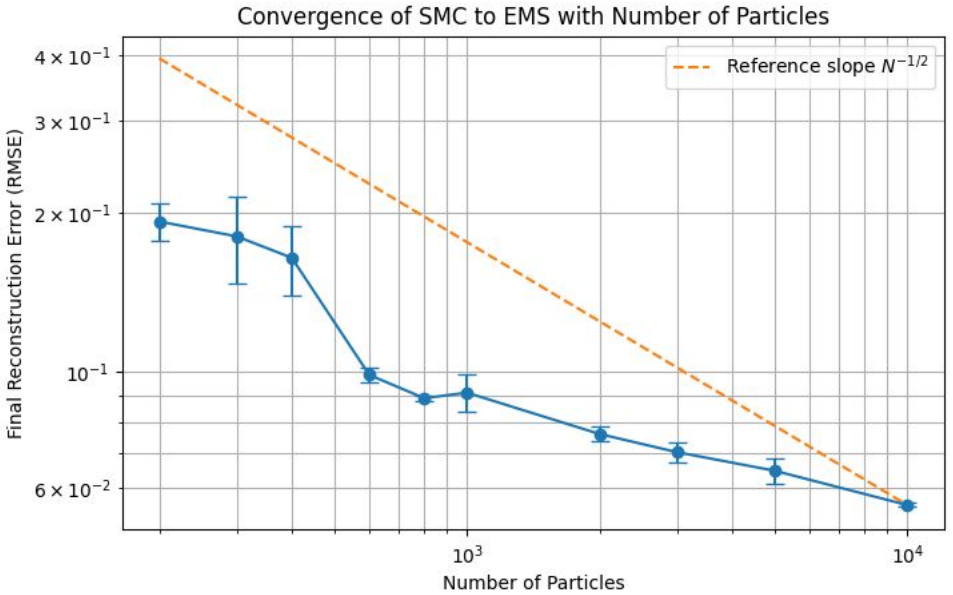
Convergence of SMC towards EMS

Proposition 3 states that

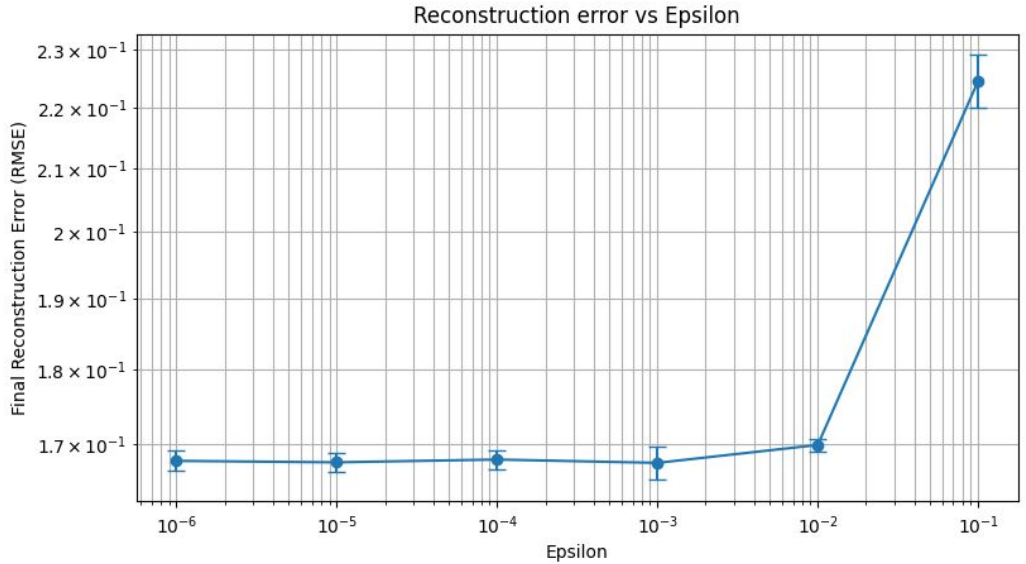
$$\mathbb{E}\left[|\hat{h}_n^N(y) - \hat{h}_n(y)|\right] \leq C_n \frac{\|g(y|x)\|_\infty}{\sqrt{N}}$$

and therefore

$$\mathrm{RMSE}(\hat{h}_n^N - \hat{h}_n) = O(\tfrac{1}{\sqrt{N}})$$



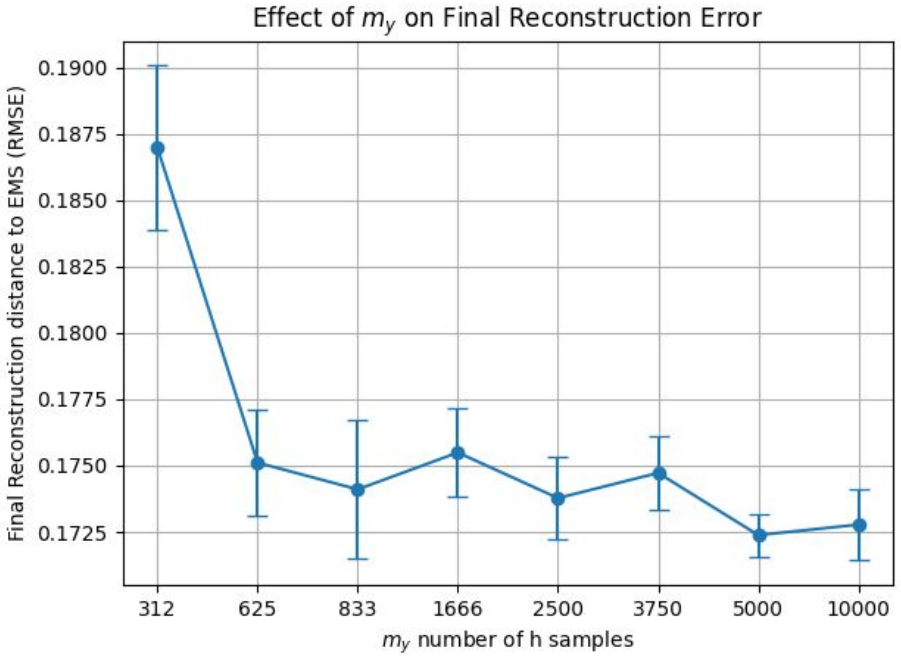
Influence of epsilon



Influence of integral approximation

What if we use less than N particles to approximate the integral of G_n against h ?

What if we use more…



PET Reconstruction via SMC

Reconstruct an image from noisy Radon data

$$h(\phi, \xi) = \int_{\mathbb{X}} \mathcal{N}\left(x\cos(\phi) + y\sin(\phi) - \xi; 0, \sigma^2\right) f(x, y)\mathrm{d}x\mathrm{d}y$$



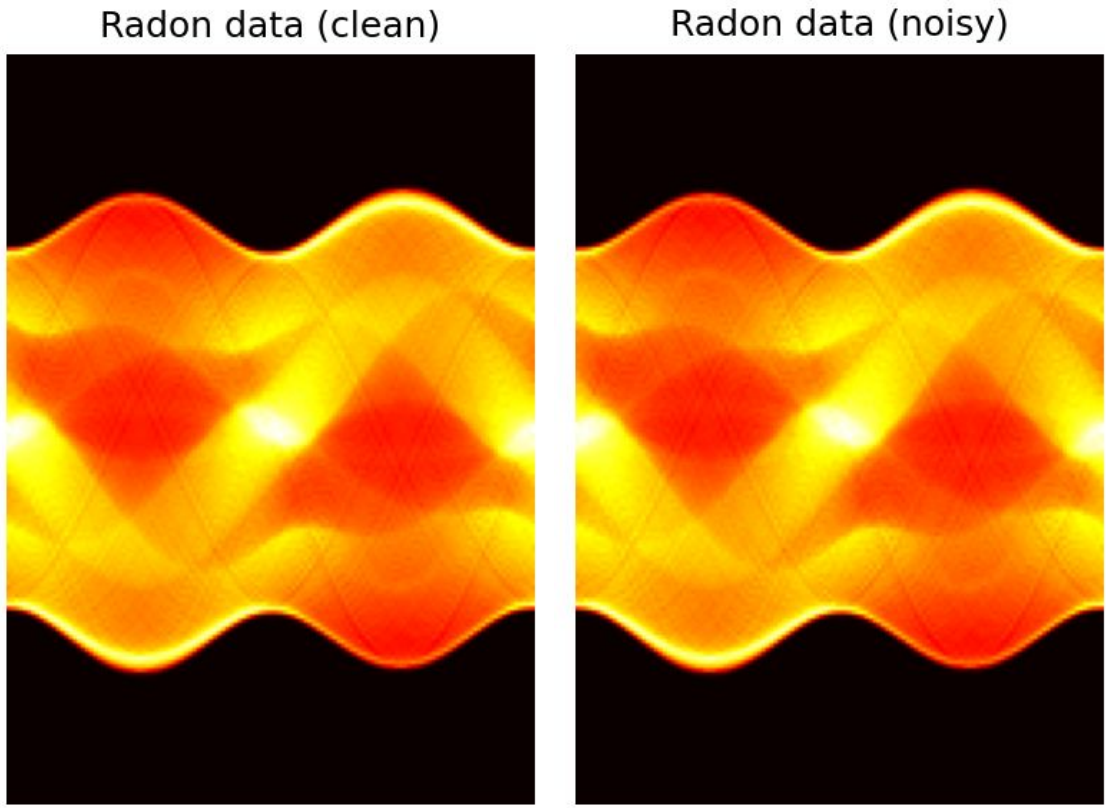
Radon Data & Poisson Noise

Resolution : 128 * 128

Number of particles : 20 000

Poisson scale : 5e4

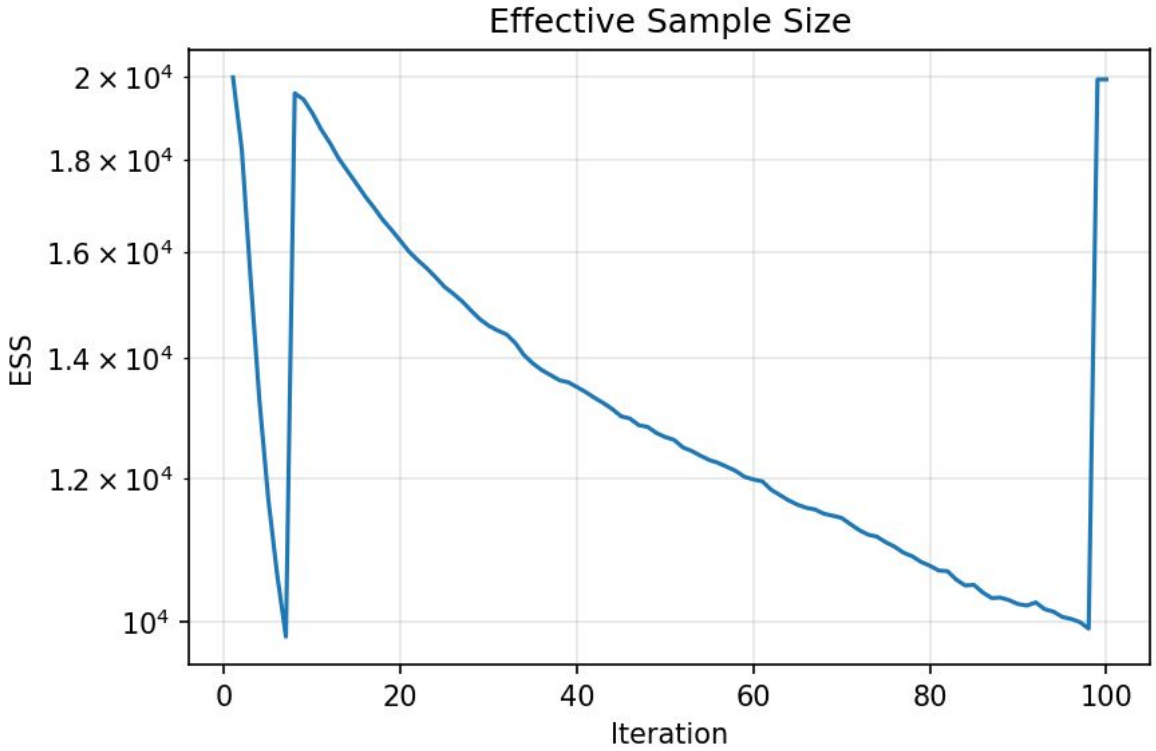
Sigma : 0.02



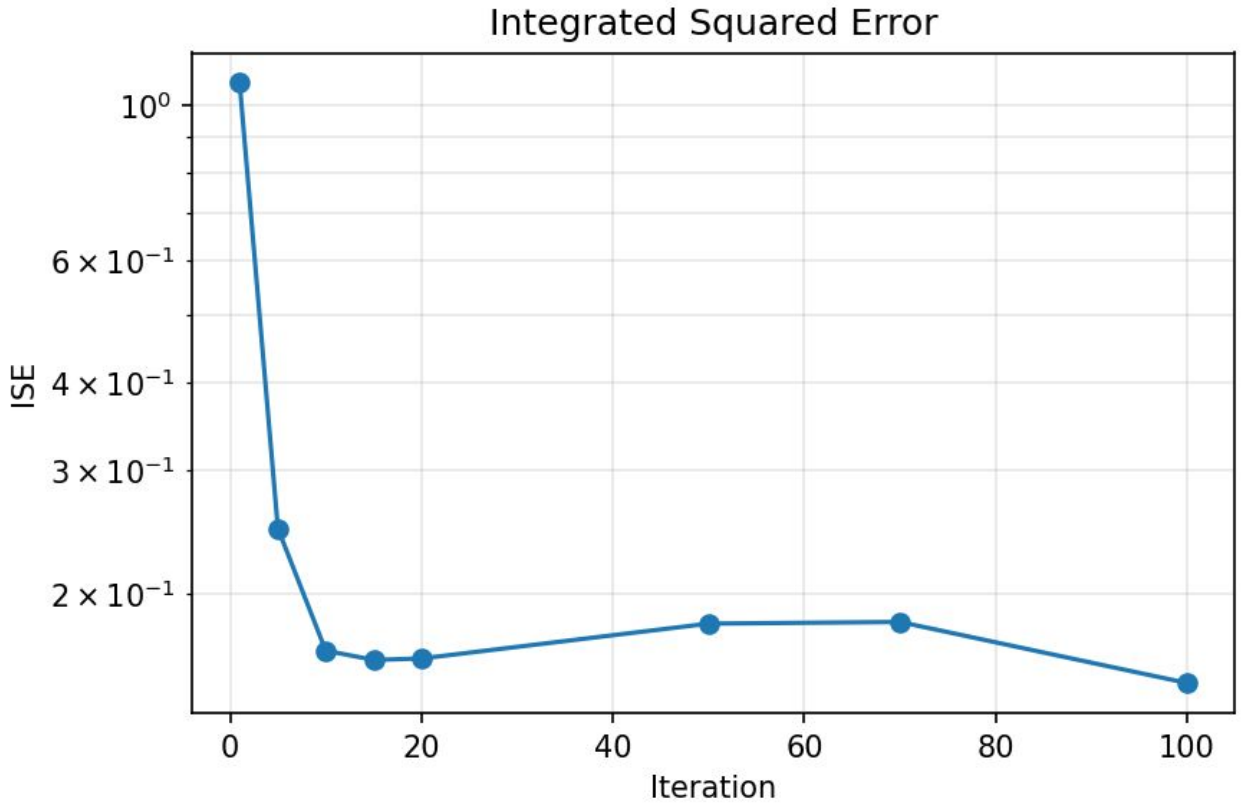
SMC algorithm : ESS curve

Multinomial Resampling

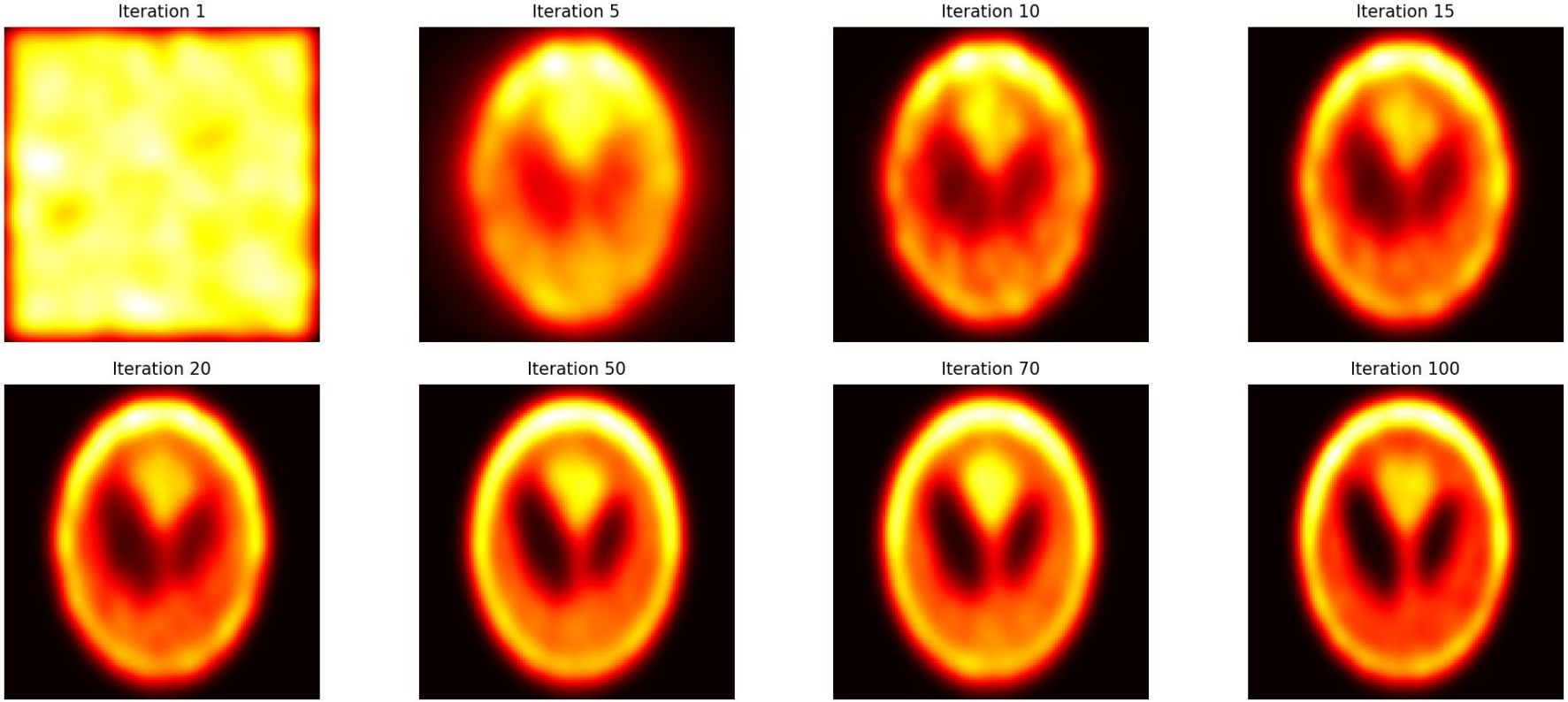
when ESS < N/2



SMC algorithm : Reconstruction Quality



SMC algorithm : Final Results



SMC algorithm : Final Results

Computational time : 1 hour for 100 iterations and 20 000 particles

Final ESS : 19 945

Final ISE : 0.149